A Complete Method for Checking Hurwitz Stability of a Polytope of Matrices *

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Abstract: We present a novel method for checking the Hurwitz stability of a polytope of matrices. First we prove that the polytope matrix is stable if and only if two homogenous polynomials are positive on a simplex, then through a newly proposed method, i.e., the weighted difference substitution method, the latter can be checked in finite steps. Examples show the efficiency of our method.

Key words: polytope of matrices; Hurwitz stability

1 Introduction

Given a linear time-invariant system $\dot{x}(t) = Ax(t)$, it is asymptotically stable if the system matrix A is Hurwitz stable, i.e., all eigenvalues of A have negative real parts. Sometimes, we need to consider a type of system uncertainty in which case the family of system matrices forms a polytope, i.e., A is varying in

$$\mathbf{A} = \left\{ \sum_{k=1}^{m} q_k A_k : \sum_{k=1}^{m} q_k = 1, q_k \geq 0 \text{ for all k} \right\}, \tag{1}$$

where $A_1, \ldots, A_m \in \mathbb{R}^{n\times n}$ are constant matrices. We say that $\mathbf{A}$ is robustly Hurwitz stable if each matrix in $\mathbf{A}$ is Hurwitz stable. The stability of a polytope of matrices cannot be derived from the stability of all its edges [1], which is the case of the stability of a polytope of polynomials [2]. In fact, [3] proved that for a polytope of $n \times n$ matrices, the stability of all $2n-4$ dimensional faces can guarantee the stability of the polytope, and the number $2n-4$ is minimal. But checking the stability of $2n-4$ dimensional faces of a polytope is also a difficult task. For a matrix polytope with normal vertex matrices, [4] proved that

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the stability of vertex matrices is necessary and sufficient for the stability of the whole polytope. Meanwhile, serval sufficient criterions [5–11] are provided to check stability of matrix polytopes.

Based on the newly proposed results on checking positivity of forms (i.e., homogenous polynomials), we present a method for checking the stability of a polytope of matrices in this paper, this method is complete, moreover, it only has a power exponential complexity.

2 Notations

- $\mathbb{R}$: the field of real numbers.
- $\mathbb{Z}$: the set of all integers.
- $\mathbb{N}$: the set of all nonnegative integers.
- $\mathbb{R}^{m\times n}$: the space of $m \times n$ real matrices.
- I_n: the identity matrix of order n.
- $\det A$: the determinant of a square matrix A.
- Δ_f: the Hurwitz matrix of the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} \ldots + a_0$, it is an $n \times n$ matrix defined as

$$\Delta_f = \begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & 0 \\ 0 & a_n & a_{n-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

  The successive principal minors of Δ_f are denoted by $\Delta_k, k = 1, 2, \ldots, n$.

- $(k_1 k_2 \ldots k_m)$: a permutation of $\{1, 2, \ldots, m\}$, which changes i to $k_i, i = 1, \ldots, m$.
- Θ_m: the set of all $m!$ permutations of $\{1, 2, \ldots, m\}$.
- $\deg(f)$: the degree of a polynomial.
- A^T: the transpose of a matrix or vector A.
- S_m: the $m - 1$ dimensional simplex in $\mathbb{R}^m$, i.e.,

$$S_m = \{(x_1, \ldots, x_m) : \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, \ldots, m\}.$$

- $[x]$: the largest integer not exceeding the number x.

3 Main Results

Suppose

$$A = \sum_{k=1}^{m} q_k A_k$$

is a matrix in $\mathbf{A}$, denote its characteristic polynomial by

$$f_A(s) \triangleq \det(sI_n - A) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0. \tag{2}$$

It is well known that, for all $1 \le i \le n$, $(-1)^i a_{n-i}$ equals the sum of all principle minors of order i of A, hence a_{n-i} is a form of degree i on $q_1, q_2, \ldots, q_m$. Denote by b_{ij} the (i,j)th entry of the Hurwitz matrix of $f_A(s)$, then $b_{ij} = 0$, or b_{ij} is a form of degree $2j - i$ on $q_1, q_2, \ldots, q_m$. Since

$$\Delta_k = \sum_{(j_1 j_2 \ldots j_k) \in \Theta_k} \pm b_{1j_1} b_{2j_2} \cdots b_{kj_k},$$

and

$$b_{1j_1} b_{2j_2} \cdots b_{kj_k} = 0,$$

or

$$\deg(b_{1j_1} b_{2j_2} \cdots b_{kj_k}) = \sum_{i=1}^{k} (2j_i - i) = \sum_{i=1}^{k} i = \frac{k(k+1)}{2},$$

we can see that Δ_k is a form of degree $k(k+1)/2$ on $q_1, q_2, \ldots, q_m$.

Theorem 1. *There exists a Hurwitz stable matrix in the matrix polytope* $\mathbf{A}$*, then* $\mathbf{A}$ *is Hurwitz stable if and only if*

$$\Delta_{n-1}(q_1, \ldots, q_m) > 0 \text{ and } a_0(q_1, \ldots, q_m) > 0, \quad (q_1, \ldots, q_m) \in S_m. \tag{3}$$

Proof. The necessity is directly from Routh-Hurwitz criterion. If $\mathbf{A}$ is not Hurwitz stable, then by continuity there must exist a matrix A in $\mathbf{A}$ which has eigenvalues lying on the imaginary axis. Suppose eigenvalues of A are $s_1, \ldots, s_n$. If some s_i equals zero, then

$$a_0 = (-1)^n s_1 s_2 \cdots s_n = 0,$$

which contradicts the hypothesis $a_0 > 0$. If some s_i and s_j are a pair of conjugate eigenvalues of A on the imaginary axis, then from Orlando's formula [12],

$$\Delta_{n-1} = (-1)^{\frac{n(n-1)}{2}} \prod_{1 \le i < j \le n} (s_i + s_j) = 0,$$

which contradicts the hypothesis $\Delta_{n-1} > 0$. $\square$

Remark: [5] also proved that a polytope matrix $\mathbf{A}$ is robustly stable if and only if an associated form $\gamma \det \widetilde{A}$ is positive on a simplex, where $\widetilde{A}$ is the Kronecker sum of the matrix A in $\mathbf{A}$ with itself, and γ is the sign of $\det \widetilde{A}$. $\widetilde{A}$ is an $n^2 \times n^2$ matrix, hence the degree of $\gamma \det \widetilde{A}$ is much larger than Δ_{n-1} and a_0 in (3).

A newly proposed method, i.e. the difference substitution method [13,14], can be used to check positivity of forms efficiently. [15] proved that a form is positive if and only if we can get forms with positive coefficients after finite steps of varied forms of substitutions, i.e., weighted difference substitutions(WDS). [16] further gave a bound for the number of steps required, and pointed out that the WDS method is complete in checking positivity or nonnegativity of integral forms. We will introduce this method more detailedly in Section 4. Based on this method, we have

Theorem 2. *Suppose entries of $A_k, k = 1, \dots, m$ in (1) are all rational, the magnitudes of coefficients of Δ_{n-1} and a_0 in (3) are bounded by M, then the Hurwitz stability of $\mathbf{A}$ can be checked by an algorithm with complexity*

$$O\left(m^{m+1}n^{2m^2}(n^{2m}\ln M + n^{2(m+1)}\ln m + 2(m^2 + mn^{2m})\ln n)\right).$$

4 Positivity of forms on simplices

In this section, we will introduce the WDS method for checking positivity of forms [16], and analyze the complexity of checking Hurwitz stability of matrix polytopes through this method.

Suppose $\theta = (k_1 k_2 \dots k_m) \in \Theta_m$, let $P_\theta = (p_{ij})_{m\times m}$ be the permutation matrix corresponding θ, that is

$$p_{ij} = \begin{cases} 1, & j = k_i \\ 0, & j \neq k_i \end{cases}.$$

Given $T_m \in \mathbb{R}^{m\times m}$, where

$$T_m = \begin{pmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{m} \\ 0 & \frac{1}{2} & \dots & \frac{1}{m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{m} \end{pmatrix}, \tag{4}$$

let

$$A_\theta = P_\theta T_m,$$

and call it the WDS matrix determined by the permutation θ. The variable substitution $\mathbf{x} = A_\theta \mathbf{y}$ corresponding θ is called a WDS, where $\mathbf{x} = (x_1, x_2, \dots, x_m)^T, \mathbf{y} = (y_1, y_2, \dots, y_m)^T$.

Let $f(\mathbf{x}) \in \mathbb{R}[x_1, x_2, \dots, x_m]$ be a form, we call

$$\mathrm{WDS}(f) = \bigcup_{\theta\in\Theta_m} \{f(A_\theta \mathbf{x})\} \tag{5}$$

the WDS set of f,

$$\mathrm{WDS}^{(k)}(f) = \bigcup_{\theta_k \in \Theta_m} \cdots \bigcup_{\theta_1 \in \Theta_m} \{f(A_{\theta_k} \cdots A_{\theta_1} \mathbf{x})\} \tag{6}$$

the kth WDS set of f for positive integer k, and set $\mathrm{WDS}^{(0)}(f) = \{f\}$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. For a form of degree d

$$f(x_1, x_2, \ldots, x_m) = \sum_{|\alpha|=d} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m},$$

if all coefficients c_α are nonzero, we say f has complete monomials.

It is obvious that if there exists $k \in \mathbb{N}$, such that forms in $\mathrm{WDS}^{(k)}(f)$ all have complete monomials, and their coefficients are all positive, then f is positive on S_m. In fact, the reverse is also true, and for integral forms, the upper bound for k can also be estimated.

Theorem 3 ([16]). *Suppose $f \in \mathbb{Z}[x_1, x_2, \ldots, x_m]$ is a form of degree d, and the magnitudes of its coefficients are bounded by M, then f is positive on S_m, if and only if there exists $k \le C_p(M, m, d)$, such that each form in $\mathrm{WDS}^{(k)}(f)$ has complete monomials, and its coefficients are all positive, where*

$$C_p(M, m, d) = \left\lceil \frac{\ln\left(2^{d^m} M^{d^m+1} m^{d^{m+1}+d} d^{(m+1)d+md^m} (d+1)^{(m-1)(m+2)}\right)}{\ln m - \ln(m-1)} \right\rceil + 2 \tag{7}$$

Remark: The $C_p(M, m, d)$ in (7) provides a theoretical upper bound of the number of steps of substitutions required to check positivity of an integral form. In practice, numbers of steps used are generally much smaller than this bound [15].

proof of Theorem 2. A form $f(q_1, \ldots, q_m)$ of degree d has at most

$$\binom{d+m-1}{m-1} \le (d+1)^{m-1}$$

monomials thus the number of arithmetic operations of computing $\mathrm{WDS}(f)$ is bounded by

$$m!(d+1)^{m-1}(d+1)^{m(m-1)} \le m^m (d+1)^{m^2}. \tag{8}$$

Moreover

$$C_p(M, m, d) = O\left(m(d^m \ln M + d^{m+1} \ln m + (m^2 + md^m) \ln d)\right),$$

and $\deg(\Delta_{n-1}) = n(n-1)/2, \deg(a_0) = n$, therefore the complexity of our method for checking the robust Hurwitz stability of a polytope of $n \times n$ matrices is

$$O\left(m^{m+1} n^{2m^2} (n^{2m} \ln M + n^{2(m+1)} \ln m + 2(m^2 + mn^{2m}) \ln n)\right).$$

$\square$

5 Examples

First we will illustrate our method through an example from [1]. Suppose $\mathbf{A}$ is a polytope of following matrices

$$A_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0.1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0.1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 0.1 \end{pmatrix},$$

let $A = q_1A_1 + q_2A_2 + q_3A_3$, and denote by $f_A(s)$ the characteristic polynomial of A. The second successive principle minor of the Hurwitz matrix of $f_A(s)$ is

$$\begin{aligned}\Delta_2 =& \frac{63}{25}\,{q_1}^3 + \frac{99}{25}\,{q_1}^2 q_3 + \frac{243}{50}\,{q_3}^2 q_1 + \frac{144}{25}\,q_1 {q_2}^2 + \frac{153}{25}\,q_1 q_2 q_3 \\ &+ \frac{144}{25}\,{q_1}^2 q_2 + \frac{243}{50}\,q_2 {q_3}^2 + \frac{63}{25}\,{q_2}^3 + \frac{99}{25}\,{q_2}^2 q_3 + \frac{171}{50}\,{q_3}^3,\end{aligned}$$

which is obviously positive on S_3. The constant term of $f_A(s)$ is

$$\begin{aligned}a_0 =& \frac{9}{10}\,{q_1}^3 + \frac{9}{10}\,{q_2}^3 - \frac{23}{5}\,q_1 q_2 q_3 - \frac{13}{10}\,{q_1}^2 q_3 + \frac{7}{10}\,{q_1}^2 q_2 \\ &- 3/10\,q_2 {q_3}^2 - 3/10\,{q_3}^2 q_1 + \frac{7}{10}\,q_1 {q_2}^2 - \frac{13}{10}\,{q_2}^2 q_3 + \frac{19}{10}\,{q_3}^3.\end{aligned} \tag{9}$$

a_0 is not positive on S_3 since the following form (with a difference of a positive constant factor) belongs to $\mathrm{WDS}^{(3)}(a_0)$ and its coefficients are all negative:

$$\begin{aligned}&- 6516\,q_1 q_2 q_3 - 1296\,{q_1}^3 - 891\,{q_2}^3 - 3888\,{q_1}^2 q_3 - 3888\,{q_1}^2 q_2 - 1568\,q_2 {q_3}^2 \\ &- 2828\,{q_3}^2 q_1 - 3483\,q_1 {q_2}^2 - 2223\,{q_2}^2 q_3 - 236\,{q_3}^3,\end{aligned}$$

therefore the polytope $\mathbf{A}$ is not Hurwitz stable.

Furthermore, we have checked 900 polytopes of matrices for $n = 2, 3, 4$ and $m = 2, 3, 4$, i.e., 100 polytopes for each pair (n, m). The vertexes of these polytopes are generated following a similar method as was described in [11]: their entries are real numbers with 4 significant numbers and uniformly distributed in the interval $[-1, 1]$, moreover the maximal real parts of their eigenvalues equal -0.0001 (if not so, a shift should be performed). Table

Table 1: Time used to check robust Hurwitz stability of 100 polytopes for each pair (n, m)

n	m	number of stable / unstable polytopes	total time (in seconds)
2	2	67 / 33	0.125
	3	29 / 71	1.578
	4	11 / 89	2537.360
3	2	58 / 42	0.123
	3	28 / 72	2.009
	4	9 / 91	665.129
4	2	54 / 46	0.090
	3	21 / 79	2.608
	4	4 / 96	1894.602

1 shows the time used to check the stability of these polytopes on a computer equipped with Intel Core 2 Duo E4500 CPU at 2.2 GHz and 4.0 GB of RAM memory, our program have been implemented in the computer algebra system Maple.

Remark: Generally, the time gets longer with the increase of n or m, but there are some extreme examples that very long time may be spent even for small n and m. For example, in our experiment, it takes 1129.656 and 1172.234 seconds respectively to check the Hurwitz stability of two matrix polytopes generated for $(n, m) = (2, 4)$, this makes the time corresponding to $(n, m) = (2, 4)$ much longer than that corresponding to $(n, m) = (3, 4)$ or $(n, m) = (4, 4)$ in Table 1.

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